SOLUÇÃO GLOBAL E ESTABILIDADE ASSINTÓTICA PARA A EQUAÇÃO DE ONDA COM DISSIPAÇÃO NÃO LINEAR NA FRONTEIRA E TERMO FONTE

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Resumo- Estudamos a existência global e taxas de decaimento uniforme de soluções do seguinte problema

$$(P) \qquad \begin{cases} u_{tt} - \Delta u = |u|^\rho u \text{ em } \Omega\times]0,+\infty[\\ u = 0 \text{ sobre } \Gamma_0\times]0,+\infty[\\ \partial_\nu u + g(u_t) = 0 \text{ sobre } \Gamma_1\times]0,+\infty[\\ u(x,0) = u^0(x), \ u_t(x,0) = u^1(x); \end{cases}$$

onde Ω é um domínio limitado do $\mathbf{R}^n$, $n \geq 1$, com fronteira $\Gamma = \Gamma_0 \bigcup \Gamma_1$ regular e $0 < \rho < \frac{2}{n-2}$, $n \geq 3$; $\rho > 0$, $n = 1, 2$.

Palavras-chave: Equação de onda, feedback fronteira, termo fonte.

GLOBAL SOLVABILITY AND ASYMPTOTIC STABILITY FOR THE WAVE EQUATION WITH NONLINEAR BOUNDARY DAMPING AND SOURCE TERM

Abstract- We study the global existence and uniform decay rates of solutions of the following problem

$$(P) \qquad \begin{cases} u_{tt} - \Delta u = |u|^\rho u \text{ in } \Omega\times]0,+\infty[\\ u = 0 \text{ on } \Gamma_0\times]0,+\infty[\\ \partial_\nu u + g(u_t) = 0 \text{ on } \Gamma_1\times]0,+\infty[\\ u(x,0) = u^0(x), \ u_t(x,0) = u^1(x); \end{cases}$$

where Ω is a bounded domain of $\mathbf{R}^n$, $n \geq 1$, with a smooth boundary $\Gamma = \Gamma_0 \bigcup \Gamma_1$ and $0 < \rho < \frac{2}{n-2}$, $n \geq 3$; $\rho > 0$, $n = 1, 2$.

KeyWord: Wave equation, boundary feedback, source term.

1 INTRODUÇÃO

This paper is concerned with the existence and uniform decay rates of solutions of the wave equation with a source term and subject to nonlinear boundary damping

$$\begin{cases} u_{tt} - \Delta u = |u|^\rho \, u \quad \text{in } \Omega \times (0,+\infty) \\ u = 0 \quad \text{on} \quad \Gamma_0 \times (0,+\infty) \\ \partial_\nu u + g(u_t) = 0 \quad \text{on} \quad \Gamma_1 \times (0,+\infty) \\ u(x,0) = u^0(x); \quad u_t(x,0) = u^1(x), \quad x \in \Omega \end{cases} \tag{1. 1}$$

where Ω is a bounded domain of $\mathbf{R}^n$, $n \geq 1$, with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint and ν represents the unit outward normal to Γ.

Problems like (1. 1), more precisely,

$$\begin{cases} u_{tt} - \Delta u = -f_0(u) \quad \text{in } \Omega \times (0,+\infty) \\ u = 0 \quad \text{on} \quad \Gamma_0 \times (0,+\infty) \\ \partial_\nu u = -g(u_t) - f_1(u) \quad \text{on} \quad \Gamma_1 \times (0,+\infty) \\ u(x,0) = u^0(x); \quad u_t(x,0) = u^1(x), \quad x \in \Omega \end{cases} \tag{1. 2}$$

were widely studied in the literature, mainly when $f_1 = 0$, see [6, 13, 22] and a long list of references therein. When $f_0 \neq 0$ and $f_1 \neq 0$ this kind of problem was well studied by Lasiecka and Tataru [15] for a very general model of nonlinear functions $f_i(s)$, $i = 0, 1$, but assuming that $f_i(s)s \geq 0$, that is, f_i represents, for each i, an attractive force. When $f_i(s)s \leq 0$ as in the present case, the situation is more delicate, since we have a source term and the solutions can blow up in finite time. To summarize the results presented in the literature we start by the works of Levine, Payne and Smith [17, 19]. In [17] the authors considered $f_0 = 0, g = 0$, $f_1(s) = -|s|^{p-2}s$ and proved blow up of solutions when the initial energy is negative while in [19] global existence for solutions is proved when, roughly, the initial data are small and $2 < p < 2^* = \frac{2(n-1)}{n-2}$ $(n \geq 3)$ is the critical value of the trace-Sobolev imbedding $H^1(\Omega) \hookrightarrow L^p(\partial\Omega)$.

When $f_0 = 0, g(s) = s$ and $f_1(s) = -|s|^\gamma s$, Vitillaro [31] proved blow-up phenomenon, when, roughly speaking, the initial datum u^0 is large enough and u_1 is not too large, so that all possible initial data with negative energy are considered. In Vitillaro [30] the author considers, as in the present paper, the wave equation subject to a source term acting in the domain and nonlinear boundary feedback and he shows existence of global solutions as well as the blow up of weak solutions in finite time. More recently, when $f_0 = 0, g(s) = -|s|^{m-2}s$ and $f_1(s) = |s|^{p-2}s$, Vitillaro [32] proved local existence of weak solutions when $m > \frac{2^*}{2^*+1-p}$ and global existence of weak solutions when $p \geq m$ or the initial data are inside the potential well associated to the stationary problem. However, in [30, 31, 32] no decay rate of the energy is proved. It is worth mentioning other papers in connection with the so called stable set (the Potential Well) developed by Sattinger [27] in 1968, namely, [7, 10, 12, 24, 29] and references therein.

Concerning the wave equation with source and damping terms

$$u_{tt} - \Delta u + g(u_t) = f(u) \quad \text{in } \Omega \times (0, +\infty)$$

where Ω is a bounded domain of $\mathbf{R}^n$ with smooth boundary Γ or Ω is replaced by the entire $\mathbb{R}^n$, it is important to cite the works of Georgiev and Todorova [8], Levine and Serrin [18], Serrin, Todorova and Vitillaro [26] and Ikehata [9]. All the above mentioned works which involve source and damping terms, except for Cavalcanti, Domingos Cavalcanti and Martinez [4], are marked by the following feature: the damping term possesses a polynomial growth near zero.

2 RESULTADOS

We start this section by setting the inner products and norms

$$(u,v) = \int_\Omega u(x)v(x)\,dx; \quad (u,v)_{\Gamma_1} = \int_{\Gamma_1} u(x)v(x)\,d\Gamma,$$
$$||u||_p^p = \int_\Omega |u(x)|^p\,dx, \quad ||u||_{\Gamma_1,p}^p = \int_{\Gamma_1} |u(x)|^p\,d\Gamma.$$

Consider the Hilbert space

$$H^1_{\Gamma_0}(\Omega) = \left\{v \in H^1(\Omega);\, v = 0 \,\text{on}\, \Gamma_0\right\} \qquad (2.\ 1)$$

and suppose that

$(H.1)$ $0 < \rho < 2/(n-2)$ if $n \geq 3$ and $\rho > 0$ if $n = 1, 2$.

The energy associated to problem (P) is given by

$$E(t) = \tfrac{1}{2}||u_t(t)||_2^2 + J(u(t)); \quad u \in H^1_{\Gamma_0}(\Omega). \ (2.\ 2)$$

where

$$J(u) = \frac{1}{2}||\nabla u||_2^2 - \frac{1}{\rho+2}||u||_{\rho+2}^{\rho+2}; \quad u \in H^1_{\Gamma_0}(\Omega) \ (2.\ 3)$$

So, we are in a position to consider general hypotheses.

(A.1) Assumptions on g.

Consider $g : \mathbf{R} \to \mathbf{R}$ a monotone nondecreasing continuous function such that $g(0) = 0$ and satisfying the growth condition

$$(H.2) \qquad C_1|s| \leq |g(s)| \leq C_0|s|, \quad |s| > 1,$$

where C_0, C_1 are positive constants.

In order to obtain the global existence for regular solutions, the following assumptions are made on the initial data.

(A.2) Assumptions on the Initial Data.

Assume that

$$(H.3) \qquad \left\{u^0, u^1\right\} \in D(A)$$

where $D(A)$ is defined by

$$D(A) = \left\{(u,v) \in H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega); u + \mathcal{N}g(\gamma_0 v) \in D(-\Delta)\right\}$$

and

$$A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ -\Delta(u + \mathcal{N}g(\gamma_0 v)) \end{pmatrix}.$$

Here,

$$D(-\Delta) = \left\{v \in H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega); \partial_\nu v = 0 \quad \text{on } \Gamma_1\right\},$$

and $\mathcal{N} : H^{-1/2}(\Gamma_1) \longrightarrow H^1_{\Gamma_0}(\Omega)$ is the Neumann map defined by

$$\mathcal{N}p = q \Leftrightarrow \begin{cases} -\Delta q = 0 & \text{in } \Omega \\ q = 0 & \text{on } \Gamma_0 \\ \partial_\nu q = p & \text{on } \Gamma_1. \end{cases}$$

Taking into account the operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$, where $\mathcal{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ and considering $U = \begin{pmatrix} u \\ u' \end{pmatrix}$, we can rewrite problem (P) as

$$\frac{dU}{dt} + AU = \begin{pmatrix} 0 \\ |u|^\rho u \end{pmatrix}.$$

Let us define the following sets:

$$A = \{(\lambda, E) \in [0, +\infty) \times \mathbb{R}; f(\lambda) \leq E < d; \quad \lambda < \lambda_1\},$$

$$B = \{(\lambda, E) \in [0, +\infty) \times \mathbb{R}; f(\lambda) \leq E < d; \quad \lambda > \lambda_1\},.$$

We have that:

(i) If $(||\nabla u^0||_2, E(0)) \in A$, then weak solutions possess an extension to the whole interval $[0, +\infty)$.

(ii) If $(||\nabla u^0||_2, E(0)) \in B$, then weak solutions blow up in finite time.

We observe that, in both cases, the initial energy $E(0) < d$ and it can also be negative. So, *one is not expected to obtain global solutions*, even if $E(0) < d$, in view of the item (ii) above. A natural question arises in this context: It would be possible to obtain global solutions if $E(0) \geq d$? This question is very difficult to be answered in a general setting. However, if we assume that: There exists $t_0 \in [0, t_{max})$ such that

$$(H.4) \qquad E(u(t_0)) < d \quad \text{and} \quad ||\nabla u(t_0)||_2 < \lambda_1,$$

the answer is positive.

However, under the assumption that there exists $t_0 \in [0, t_{max})$ such that

$$(H.5) \qquad E(u(t_0)) < d \quad \text{and} \quad ||\nabla u(t_0)||_2 > \lambda_1,$$

solutions will blow up in finite time.

Now we are in a position to state our main results.

Theorem 2.1 *Under assumptions (H.1)-(H.4), problem (1. 1) possesses a unique regular solution u in the class*

$$u \in L^\infty_{loc}\left(0,\infty; H^1_{\Gamma_0}(\Omega)\right), \; u' \in L^\infty_{loc}\left(0,\infty; H^1_{\Gamma_0}(\Omega)\right), \\ u'' \in L^\infty_{loc}\left(0,\infty; L^2(\Omega)\right) \qquad (2.\ 4)$$

and $||\nabla u(t)||_2 < \lambda_1$ *for all* $t \geq t_0$. *Furthermore, the energy* $E(t)$ *given by*

$$E(t) = \frac{1}{2}\left|\left|u'(t)\right|\right|_2^2 + \frac{1}{2}\left|\left|\nabla u(t)\right|\right|_2^2 - \frac{1}{\rho+2}\left|\left|u(t)\right|\right|_{\rho+2}^{\rho+2}, \qquad (2.\ 5)$$

has the following decay rate

$$E(t) \leq S\left(\frac{t}{T_0} - 1\right)E(0), \qquad (2.\ 6)$$

for all $t \geq T_0 > 0$,*where* $S(t)$ *is the solution of the following differential equation*

$$S'(t) + q\left(S(t)\right) = 0$$

and q is a strictly increasing function.

Theorem 2.2 *Let the initial data belong to* $H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ *satisfying* $(H.4)$ *such that the same hypotheses on g and ρ hold. Then, problem (1.1) possesses a unique weak solution in the class*

$$u \in C^0([0,\infty), H^1_{\Gamma_0}(\Omega)) \cap C^1([0,\infty); L^2(\Omega)), \qquad (2.\ 7)$$

and $||\nabla u(t)||_2 < \lambda_1$ *for all* $t \geq t_0$. *Besides, the weak solution has the same decay given in (2. 6).*

3 CONCLUSÕES

In the present paper we generalize substantially the results given in [4] in the following sense: (i) g is considered just a monotone nondecreasing continuous function while in [4] this function is assumed to be of $C^1(\mathbb{R})$; (ii) No growth assumption is imposed on the function g near the origin, while in [4] the authors consider $|g_0(s)| \leq |g(s)| \leq |g_0^{-1}(s)|$, $s \in [-1,1]$, where g_0 is a strictly increasing and odd function of class C^1; (iii) The following assumption is considered: There exists $t_0 \in [0, t_{max})$ such that

$$E(u(t_0)) < d \quad \text{and} \quad ||\nabla u(t_0)||_2 < \lambda_1,$$

while in [4] the same assumption was considered in the particular case when $t_0 = 0$; (iv) Setting $h(x) := x - x^0$, $x^0 \in \mathbb{R}^n$, the unique geometric condition is assumed on the uncontrolled portion of the boundary Γ_0: $h \cdot \nu \leq 0$, on Γ_0, while in [4], in spite of the same condition is also assumed on Γ_0, the restrictive geometrical condition is imposed on Γ_1: $h \cdot \nu > 0$ on Γ_1.

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