A ESTABILIDADE NA PRESENÇA DE UMA VARIEDADE INVARIANTE E A SUA APLICAÇÃO A ESTABILIÇABILIDADE DE SISTEMAS DE CONTROLE NÃO LINEARES

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Resumo- De acordo com a suposição da presença de uma variedade invariante de atração com respeito ao qual o equilíbrio é assintóticamente estável, é comprovado a estabilidade assintótica de um equilíbrio de um sistema não linear. Este resultado é aplicado ao problema de estabiliçabilidade de um sistema de controle.

Palavras-chave: Variedade invariante, estabiliçabilidade, controle não linear

THE STABILITY IN THE PRESENCE OF AN INVARIANT MANIFOLD AND THEIR APPLICATION TO STABILIZABILITY OF NON LINEAR CONTROL SYSTEMS

Abstract- Under the assumption of the presence of an attracting invariant manifold with respect to which the equilibrium is asymptotically stable, is proved the asymptotic stability of an equilibrium of a nonlinear system. This result is applied to the problem of stabilizability of a control system.

KeyWord: Invariant manifold, stabilizability, non linear control

1. INTRODUCTION

In many problems of engineering which have been studied in recent years, a crucial aspect is to simplify stability analysis by the use of invariant manifolds. This problem is intimately related to the decomposition of a given system into subsystems. It obviously consist of two problems: 1. To establish the existence of invariant manifolds and to find them; 2. To formulate the condition under which the system as a whole is stable on the basis of the stability behavior of the subsystems.

The first problem is a difficult one in general case, involving partial differential equations, and only relatively special cases have been treated so far (see, for instance, [10]).

As far as the second point is concerned, mainly two methods have been used in previous contributions: On the on hand, there is the centre manifold theory which has the limitation of depending strongly on the first-order approximation of the system. On the other hand, Lyapunov function techniques have been used in the case of systems in triangular form (cf. M. Vidyasagar [14]).

In the present paper, we solved the second problem for a general nonlinear system with an invariant manifold. The central result is that if the reduction of the system to the invariant manifold itself is a stable attractor, then the system as a whole is asymptotically stable. We formulate our results for systems in $\mathbb{R}^n$, but the method used is applicable equally to infinite-dimensional systems. This aspect will be presented in a future paper in which also the question of the size of the region of attraction will be treated.

The problem treated here plays a particularly significant role in control theory (see, for instance, [1-4] and [6-9]) and the theory of interconnected systems (large scaled systems), (e.g., [15]). The application considered in this paper is to the problem of stabilization of nonlinear systems. The methods used are of a kind that allows their extension beyond the scop of finite-dimensional systems and theorems of a strictly local nature, and they do not depend on any linearity assumptions.

For the sake of simplicity of presentation, we formulate our results in terms of time-invariant systems; their extension to time-varying systems is just a technical matter.

For the basic theory of Lyapunov stability in the form in which it is used here, we recommend to the reader to consult the book [11].

2. NOTATIONS AND DEFINITIONS

Let $|\cdot|$ be a norm in $\mathbb{R}^n$. The open ball of center x and radius r and the other for a set $A \subset \mathbb{R}^n$ will be denoted by

$$B(x,r) := \{y \in \mathbb{R}^n \ | \ |y-x| < r\};$$
$$B(A,r) := \cup\{B(x,r) \ | \ x \in A\}, \qquad (A \subset \mathbb{R}^n),$$

respectively.

Consider the system

$$\dot{x} = f(x) \tag{1}$$

where $x \in \mathbb{R}^n$ and f satisfies a local Lipschitz condition. We will use the notation $x(t,x^0)$ for the solution of (1) with initial condition $(0,x^0) = x_0$. We denoted by $\gamma^+(x)$ the positive semiorbit with initial point x; i.e., $\gamma^+(x) = \{y \in \mathbb{R}^n \mid \exists t \geq 0 \quad \text{such that} \quad x(t,x) = y\}$. Analogously for a set $A \subset \mathbb{R}^n$, $\gamma^+(A) = \bigcup\{\gamma^+(x) \mid x \in A\}$.

Definition 0.1. *A set $S \subset \mathbb{R}^n$ is said to be an invariant set for the system (1) if $\gamma^+(S) \subset S$.*

Given the open set $U \subset \mathbb{R}^n$, by the *restricted positive semiorbit* with initial point x, $\gamma_U^+(x)$ we mean the set $\{x(t,x) \mid t \in I_x\}$, where I_x denotes the maximal interval starting at $t = 0$ such that $x(t,x) \in U$, for all $t \in I_x$.

Definition 0.2. *The set $S \subset \mathbb{R}^n$ is a $U-$invariant set if $U \cap S \neq \varnothing$ and $\gamma_U^+(S \cap U) \subset S$ (i.e., solutions starting in $S \cap U$ do not leave S while remaining in U).*

Definition 0.3. *The $U-$invariant set S is uniformly $U-$stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$x \in U \cap B(S,\delta)$$

implies

$$\gamma_U^+(x) \subset B(S,\epsilon) \tag{2}$$

Definition 0.4. *The $U-$invariant set S is uniformly $U-$attracting if every positive semiorbit contained in U tends to S; i.e. for every $x \in U$ such that $I_x = [0,\infty)$,*

$$\lim_{t\to\infty} d(x(t,x),S) = 0$$

holds, where d is the distance function $(d(y,S)) = \inf\{|y-z| \ : \ z \in S\})$.

3. MAIN RESULTS

We will consider, the system (1) given in the form

$$\begin{aligned} \dot{x_1} &= f_1(x_1,x_2) \\ \dot{x_2} &= f_2(x_1,x_2), \end{aligned} \tag{3}$$

where $x_1 \in \mathbb{R}^m$, $x_2 \in \mathbb{R}^{n-m}$, $x = (x_1,x_2)$ and $f = (f_1,f_2)$. It will be assumed that there exists an invariant set S which is an $m-$ dimensional manifold defined by smooth function h :

$$S: \quad x_2 = h(x_1). \tag{4}$$

The invariance of S implies $x_2(t,x^0) = h(x_1(t,x^0))$ if x^0 satisfies $x_2^0 = h(x_1^0)$ (see Definition 0.1). Moreover, the invariance of S implies that h satisfies the partial differential equation

$$dh \cdot f_1(x_1,h(x_1)) = f_2(x_1,h(x_1)), \tag{5}$$

where dh is the jacobian matrix.

We observe that the restriction of system (3) to the invariant manifold S is given by the $m-$dimensional system

$$\dot{x_1} = f_1(x_1,h(x_1)), \tag{6}$$

Here we are using x_1 as coordinate on S (by assigning to the point $(x_1,h(x_1))$, x_1 as its coordinate). If we suppose $f(0) = 0$ and $h(0) = 0$, then $0 \in \mathbb{R}^n$ is an equilibrium point of (3) which is contained in S, and $x_1 = 0$ is an equilibrium point of (6).

We have the following results:

Theorem 0.5. *Let f, h be as specified above and let U be an open neighborhood of $x = 0$. We assume that S is uniformly $U-$stable and $x_1 = 0$ is an asymptotically stable equilibrium point of (6). Then $x = 0$ is a stable equilibrium point of (3).*

Theorem 0.6. *If the hypotheses of Theorem 0.5 are satisfied, and in addition S $U-$attracting, then $x = 0$ is an asymptotically stable equilibrium point of (3).*

The proofs of the theorems 0.5 and 0.6 had been outlined in the context of locally compact metric spaces (containing $\mathbb{R}^n$ as a special case) by the second author ([12],[13]) the next corollary was stated in [12] (for a proof using Lyapunov functions see Vidyasagar [14]).

Consider the system

$$\begin{aligned} \dot{x_1} &= f_1(x_1) \\ \dot{x_2} &= f_2(x_1,x_2). \end{aligned} \tag{7}$$

It is a particular case of (3) where the first equation is independent of x_2.

Corollary 0.7. *The point $x = 0$ is an asymptotically stable equilibrium point of (7) if only if $x_1 = 0$ and $x_2 = 0$ are asymptotically stable equilibrium points of*

$$\dot{x_1} = f_1(x_1) \tag{8}$$

and

$$\dot{x_2} = f_2(0,x_2) \tag{9}$$

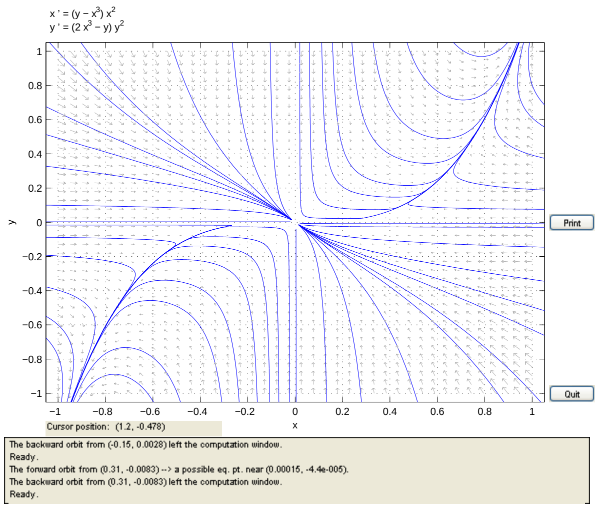
respectively.

The following example shows that this result cannot be extended along lines one might expect in analogy with the behavior of linear systems.

Example 0.8. *Consider the two-dimensional systems*

$$\begin{aligned} \dot{x_1} &= (-x_1^3 + x_2)x_1^2 \\ \dot{x_2} &= (2x_1^3 - x_2)x_2^2, \end{aligned} \tag{10}$$

which has $\{(x_1, 0) \mid x_1 \in \mathbb{R}\}$ *and* $\{(0, x_2) \mid x_2 \in \mathbb{R}\}$ *as invariant manifolds. The origin of (10) is asymptotically stable with respect to each of these invariant sets, but it is nevertheless unstable. The instability of the origin can be shown using Persidski's theory of sectors. The two regions bounded by the curves* $x_2 = 2x_1^3$ *and* $x_2 = x_1^3$ *and containing the coordinate axes in their interior, are expellers (see [11]). The configuration of the system is shown in Figure 1.*



In general, the attracting condition of the invariant manifold S is difficult to show, unless one finds an appropiated Lyapunov function. The following sufficient condition is rather restrictive but easy to verify.

Theorem 0.9. *Let system (3) with the invariant manifold* $S = \{(x_1, x_2) \mid x_2 = h(x_1)\}$ *be given. Let* U *be an open neighborhood of the equilibrium point* $x = 0$*. If for all* $(x_1, x_2) \in U \setminus S$ *the condition*

$$(x_2 - h(x_1)) \circ (f_2(x_1, x_2)) - dh \circ f_1(x_1, x_2) < 0 \tag{11}$$

is satisfied then S *is* $U-$*attracting.*

Proof. Let (y_1, y_2) be a solution of (3) in the form (7)-(8) the positive semiorbit of which is contained in U. Then it follows from (11) using the definition of y_2 and the equations (7)-(8), that

$$\frac{d}{dt}|y_2(t)^2| = 2y_2(t) \circ \dot{y_2}(t) = 2y_2(t)g_2(y_1(t), y_2(t)) < 0. \tag{12}$$

This inequality yields the attracting property in question using the usual Lyapunov type argument ($|y_2|^2$ being positive definite with respect to Y). □

4.APPLICATIONS TO CONTROL

Now we are going to present several applications of Theorem 0.6 to the problems of the stabilization of nonlinear systems by smooth state-feedback.

Consider a control system which is linear with respect to control,

$$\dot{x} = f(x) + g(x) \cdot u \tag{13}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, f and g are smooth functions which are an $n-$vector and $n \times m-$matrix, respectively, and $f(0) = 0$.

Definition 0.10. *The control system (13) is said to be stabilizable if there exists a smooth function* $u^* : \mathbb{R}^n \to \mathbb{R}^m$ *with* $u^*(0) = 0$ *which satisfies the condition that the closed-loop system*

$$\dot{x} = f(x) + g(x) \cdot u^*(x) \tag{14}$$

has the origin as a locally asymptotically equilibrium point.

a. Triangular Systems

Systems (13) is a *triangular system* if it can be written in the form

$$\begin{aligned} \dot{x_1} &= f_1(x_1) + g_1(x_1) \cdot u_1 \\ \dot{x_2} &= f_2(x_1, x_2) + g_2(x_1, x_2) \cdot u_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{x_k} &= f_k(x_1, x_2, ..., x_k) + g_k(x_1, x_2, ..., x_k) \cdot u_k \end{aligned} \tag{15}$$

where $x_1 \in \mathbb{R}^1$, $u_1 \in \mathbb{R}^{m_1}$ and $\sum_{i=1}^k n_i = n$ and $\sum_{i=1}^k m_i = n$

The next corollary is a direct consequence of Corollary 0.7 (see [14]).

Corollary 0.11. *The system (15) is stabilizable if only if each one of the systems*

$$\dot{x_i} = f_1(0, ..., 0, x_i) + g_i(0, ..., 0, x_i) \cdot u_i$$

is stabilizable.

b. Zero-dynamics

Let system (13) with an output function

$$z = c(x) \tag{16}$$

where $z \in \mathbb{R}^p$ and $c(0) = 0$, be given.

Isidori and Moog [7] introduce the concept of zero-dynamics as the internal dynamical behavior associated with the output $z(t) = 0$.

In a more formal way one defines.

Definition 0.12. *[7] If there is a manifold* M *passing through* $x = 0$*, with the properties*

i) for all $x \in M$, $c(x) = 0$ *(i.e,* $M \subset c^{-1}(0)$*),*

ii) for all $x \in M$*, there is a* $u \in \mathbb{R}^m$ *such that the vector* $f(x) + g(x) \cdot u$ *is tangent to* M*,*

iii) M *is maximal with respect to i), ii), then it will be called a zero-dynamics manifold.*

Definition 0.13. *If a zero dynamics manifold exists and there is a smooth state-feedback $u^*(x)$ defined for all $x \in M$, satisfying condition ii), we will call*

$$f^*(x) = f(x) + g(x) \cdot u^*(x) \quad (17)$$

a zero-dynamics vector field.

In the paper [7] the zero dynamics vector field was introduced for the case that the zero-feedback u^* is unique. The motive was to extend to nonlinear systems the dynamical interpretation of the zeros of invertible multivariable linear systems.

We can observe that condition ii) implies the existence of a control system defined on M, which is the restriction to M of the system (13).

The question of stabilization of (13) using the zero-dynamics concepts and Theorem 0.6, can be stated in two different forms:

1. If the restricted control system defined on M is stabilizable by $u^*(x)$, is it possible to find a smooth extension to a neighborhood of $x = 0$, of the function u^*, such that the zero-dynamics manifold turns out to be a stable attracting set of the closed-loop system ?
2. If we have a feedback control function $u^*(x)$ such that the zero-dynamics manifold M is a stable attracting set for the closed-loop system, is $z = 0$ an asymptotically stable equilibrium point of the restriction of the system to M?

These problems were solved by Byrnes and Isidori [4] for systems which have the same number m of imput and output components. We can rewrite their results using Theorem 0.6 in the following form, in which the condition on the linear part of the zero-dynamics vector field no longer appears:

Theorem 0.14. *If the $m \times m$ matrix $dc(x) \cdot g(x)$ is nonsingular at $x = 0$, then there exist a unique zero-dynamics manifold M and a unique state-feedback u^*, such that M is a stable attracting manifold of the closed-loop system (14).*

Theorem 0.15. *We assume the existence of a zero-dynamics manifold which has only one zero-dynamics vector field defined on it. Then, if $x = 0$ is an asymptotically stable equilibrium point of the zero-dynamics vector field, the system (13) is locally stabilizable.*

Theorems 0.14 and 0.15 yield the following corollary

Corollary 0.16. *If the $m \times m$ matrix $dc(x) \cdot g(x)$ is nonsingular at $x = 0$, then the systems (13) is locally stabilizable if $x = 0$ is an asymptotically equilibrium point of the zero-dynamics vector field.*

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