PROPRIEDADES SUAVIZANTES PARA A EQUACÃO KP-I

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Resumo. Neste trabalho estudamos propriedades suavizante das soluções pra uma equação de evolução dispersiva não linear bidimensional. Em particular, se o dado inicial ϕ possui certa regularidade e suficiente decaimento quando $x \to \infty$, então a solução $u(t)$ será mais regular ϕ para $0 \leq t \leq T$ onde T é o tempo de existênça da solução.

Palavras-chave: Equação KP-I, Propriedades suavizantes, Espaços de Sobolev com peso.

SMOOTHING PROPERTIES FOR THE KP-I EQUATION

Abstract- In this paper we study the smoothness properties of solutions for a bidimensional nonlinear dispersive evolution equation. In particular, if the initial data ϕ possesses certain regularity and sufficient decay as $x \to \infty$, then the solution $u(t)$ will be smoother than ϕ for $0 \leq t \leq T$ where T is the existence time of the solution.

KeyWord: KP-I equation, Smoothing properties, Weighted Sobolev space.

1. INTRODUCTION

The KdV equation is a model for water wave propagation in shallow water with weak dispersive and weak nonlinear effects. In 1970, Kadomtsev & Petviashvili [13] derived a two-dimensional analog to the KdV equation. Now known as the KP-I and KP-II equations, these equations are given by

$$u_{tx} + u_{xxxx} + u_{xx} + \epsilon u_{yy} + (uu_x)_x = 0$$

where $\epsilon = \mp 1$. In addition to being used as a model for the evolution of surface waves [1], the KP equation has also been proposed as a model for internal waves in straits or channels of varying depth and width [23], [7]. The KP equation has also been studied as a model for ion-acoustic wave propagation in isotropic media [20]. In this paper we consider smoothness properties of solutions to the KP-I equation

$$(u_t + u_{xxx} + u_x + u\,u_x)_x - u_{yy} = 0, \qquad (1)$$

$$u(x,\,y,\,0) = \phi(x,\,y). \qquad (2)$$

where $(x,\,y) \in R^2$ and $t \in R$. Certain results concerning the Cauchy problem for the KP-I equation include the following. Ukai [24] proved local well-posedness for both the KP-I and KP-II equations for initial data in $H^s(R^2)$, $s \geq 3$, while Saut [22] proved some local existence results for generalized KP equations. More recently, results concerning global well-posedness for the KP-I equation have appeared. In particular, see the works of Kenig [14] and Molinet, Saut, and Tzvetkov [19]. Here we consider the question of gain of regularity for solutions to the KP-I equation.

A number of results concerning gain of regularity for various nonlinear evolution equations have appeared. This paper uses the ideas of Cohen [3], Kato [12], Craig and Goodman [5] and Craig, Kappeler, and Strauss [6]. Cohen considered the KdV equation, showing that "box-shaped" initial data $\phi \in L^2(R^2)$ with compact support lead to a solution $u(t)$ which is smooth for $t > 0$. Kato generalized this result, showing that if the initial data ϕ are in $L^2((1+e^{\sigma x})\,dx)$, the unique solution $u(t) \in C^\infty(R^2)$ for $t > 0$. Kruzhkov and Faminskii [16] replaced the exponential weight function with a polynomial weight function, quantifying the gain in regularity of the solution in terms of the decay at infinity of the initial data. Craig, Kappeler, and Strauss expanded on the ideas from these earlier papers in their treatment of highly generlized KdV equations.

Other results on gain of regularity for linear and nonlinear dispersive equations include the works of Hayashi, Nakamitsu, and Tsutsumi [9], [10], Hayashi and Ozawa [11], Constantin and Saut [4], Ponce [21], Ginibre and Velo [8], Kenig, Ponce and Vega [15] and Vera [25], [26].

In studying propagation of singularities, it is natural to consider the bicharacteristics associated with the differential operator. For the KdV equation, it is known that the bicharacteristics all point to the left for $t > 0$, and all singularities travel in that direction. Kato [12] makes use of this uniform dispersion, choosing a nonsymmetric weight function decaying as $x \to -\infty$ and growing as $x \to \infty$. In [6], Craig, Kappeler and Strauss also make use of a unidirectional propagation of singularities in their results on infinite smoothing properties for generalized KdV-type equations for which $f_{u_{xxx}} \geq c > 0$.

For the two-dimensional case, Levandosky [17] proves smoothing properties for the KP-II equation. This result makes use of the fact that the bicharacteristics all point into one half-plane. Subsequently, in [18], Levandosky considers generalized KdV-type equations in two-dimensions, proving that if all bicharacteristics point into one half-plane, an infinite gain in regularity will occur, assuming sufficient decay at infinity of the initial data.

In this paper, we address the question regarding gain in regularity for the KP-I equation. Unlike the KP-II equation, the bicharacteristics for the KP-I equation are not restricted to a half-plane but span all of R^2. As a result, singularities may travel in all of R^2 However, here we prove that if the initial data decays sufficiently as $x \to \infty$, then we will gain a finite number of derivatives in x (as well as mixed derivatives). In order to state a special case of our gain in regularity theorem, we first introduce certain function spaces we will be using.

Definition. We define $\mathrm{X}^0(R^2) = \{u:\ u,\ \xi^3\widehat{u},\ \frac{\eta^2}{\xi}\,\widehat{u} \in L^2(R^2)\}$ equipped with the natural norm. On the space

$$\widetilde{X}^0(R^2) = \left\{ u:\ \frac{1}{\xi}\,\widehat{u}(\xi,\,\eta) \in L^2(R^2) \right\} \qquad (3)$$

we define the operator ∂_x^{-1} by $\widehat{\partial_x^{-1}u} \equiv \frac{1}{i\,\xi}\,\widehat{u}$. Therefore, in particular, we can write the norm of $X^0(R^2)$ as

$$||u||^2_{X^0(R^2)} = \int_{R^2} [\,u^2 + u^2_{xxx} + (\partial_x^{-1}u_{yy})^2\,]\,dx\,dy \qquad (4)$$

On this space of functions $X^0(R^2)$, it makes sense to rewrite e101-e102 as

$$u_t + u_{xxx} + u_x + u\,u_x - \partial_x^{-1}u_{yy} = 0, \qquad (5)$$
$$u(x,\,y,\,0) = \phi(x,\,y) \qquad (6)$$

where $(x,\,y) \in R^2$, $t \in R$ and consider weak solutions $u \in X^0(R^2)$.

Definition. Let N be a positive integer. We define the space of functions $X^N(R^2)$ as follows $\mathrm{X}^N = \{u\ :\ u \in L^2(R^2),\ \mathcal{F}^{-1}(\xi^3\,\widehat{u}) \in H^N(R^2),\ \mathcal{F}^{-1}\left(\frac{\eta^2}{\xi}\,\widehat{u}\right) \in H^N(R^2)\}$ equipped with the norm

$$||u||^2_{X^N(R^2)}$$
$$= \int_{R^2}(u^2 + \textstyle\sum_{|\alpha|\leq N}[\,(u_{xxx})^2 + (\partial_x^{-1}u_{yy})^2\,])\,dx\,dy \quad (7)$$

where $\alpha = (\alpha_1,\,\alpha_2) \in Z^+ \times Z^+$ and $|\alpha| = \alpha_1 + \alpha_2$.

Gain of Regularity Theorem. *Let u be a solution of (5)-(6) in $R^2 \times [0,T]$ such that $u \in L^\infty([0,T]; X^1(R^2))$ and*

$$\sup_{0\leq t\leq T} \int_{R^2} [u^2 + (\partial_y^L u)^2 + (1+x_+)^L(\partial_x^L u)^2]\,dx\,dy < +\infty \qquad (8)$$

for some integer $L \geq 2$. Then

$$\sup_{0\leq t\leq T} \int_{R^2} t^{|\alpha|-L}(x_+^{2L-|\alpha|-\alpha_2} + e^{\sigma x_-})(u)^2$$
$$+ \int_0^T \int_{R^2} t^{|\alpha|-L}(x_+^{2L-|\alpha|-\alpha_2-1} + e^{\sigma x_-})(u_x)^2 < \infty, \quad (9)$$

for $L+1 \leq |\alpha| \leq 2L-1$, $2L - |\alpha| - \alpha_2 \geq 1$, $\sigma > 0$ arbitrary.

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