EXISTÊNCIA DE ATRATORES PARA UM MODELO DE VIGA GENERALIZADO

Victor Rafael Cabanillas Zannini
Facultad de Ciencias Matemáticas, UNMSM, Lima-Perú, e-mail: vcabanillasz@unmsm.edu.pe

Resumo- Neste trabalho consideramos o modelo de viga não linear abstrato

$$\mu u_{tt} + \delta u_t + A^{\alpha} u + M(\|A^{\frac{\beta}{2}} u\|^2) A^{\beta} u + Lu = f$$

sobre um espaço de Hilbert separável H. Sob condições adequadas, mostramos que o sistema dinâmico $(\mathcal{H}, S_t)$ gerado pelo modelo com $f(t) \equiv f \in H$ é dissipativo e possui um atrator global $\mathcal{A}$ de dimensão fractal finita.

A questão principal do trabalho é o comportamento assintótico das soluções do problema para o caso em que as forças inerciais são pequenas respeito das forças de resistência do meio. Esta hipótese leva, formalmente, ao problema de primeira ordem

$$\delta u_t + A^{\alpha} u + M(\|A^{\frac{\beta}{2}} u\|^2) A^{\beta} u + Lu = f$$

Então, provamos que, num certo sentido, os atratores globais destes dois sistemas estão "perto"um do outro. Estimativas de decaimento exponencial para u_t and u_{tt} são obtidas.

Palavras-chave: Atrator global, sistema dissipativo, limite singular.

EXISTENCE OF ATTRACTORS FOR A GENERALIZED BEAM MODEL

Abstract- In this work, we consider the nonlinear abstract beam model

$$\mu u_{tt} + \delta u_t + A^{\alpha} u + M(\|A^{\frac{\beta}{2}} u\|^2) A^{\beta} u + Lu = f$$

on a separable Hilbert space H. We show that under suitable conditions, the dynamical system generated by this model with $f(t) \equiv f \in H$ is dissipative and possesses a global attractor $\mathcal{A}$ of a finite fractal dimension.

The main question discussed in this work is the asymptotic behaviour of the solutions of the problem for the case when the inertial forces are small with respect to the medium resistance forces. Formally, this hypothesis leads to first order problem

$$\delta u_t + A^{\alpha} u + M(\|A^{\frac{\beta}{2}} u\|^2) A^{\beta} u + Lu = f$$

Then, we prove that the global attractors of these problems are close in some sense. An exponential decay rate for u_t and u_{tt} is obtained.

KeyWord: Global attractor, dissipative system, singular limit.

1. PRELIMINARIES

In this section we give a basic theorems that we will use throughout this paper. The concepts of dissipative and asymptotically compact dynamical system, global attractor, fractal dimension, trajectories, among other concepts can be find in Chueshov (2002) and Temam (1988).

Theorem A. *Let* $(\mathbb{X}, S_t)$ *be a continuous dynamical system and* X *be a Banach space. Assume that*

(i) *there exists a continuous function* $G : X \to \mathbb{R}$ *possessing the properties*

$$\varphi_1 (\|x\|) \le G(x) \le \varphi_2 (\|x\|)$$

where $\varphi_j : \mathbb{R}^+ \to \mathbb{R}$ *are continuous functions such that* $\varphi_1 (r) \to +\infty$ when $r \to \infty$;

(ii) *there exists a derivative* $\frac{d}{dt} G(S_t)$ for $t \ge 0$ *and positive numbers* μ *and* ρ *such that*

$$\frac{d}{dt} G(S_t y) \le -\mu \ \ for \ \ \|S_t y\| > \rho$$

Then the dynamical system $(\mathbb{X}, S_t)$ *is dissipative.*

Theorem B. *Assume that a* dynamical system $(\mathbb{X}, S_t)$ *is dissipative and asymptotically compact. Let* B *be a bounded absorbing set of the system* $(\mathbb{X}, S_t)$. *Then the set* $\mathcal{A} = \omega(B)$ *is a nonempty compact set and is a global attractor of the dynamical system* $(\mathbb{X}, S_t)$. *Furthermore, the attractor* $\mathcal{A}$ *is a connected set in* X.

Theorem C. *Assume that* M *is a compact set in a Hilbert space* H. *Let* G *be a continuous mapping in* H *such that* $M \subset G(M)$. *Assume that there exists a finite dimensional projection* P *in the space* H *such that*

$$\|P(Gv_1 - Gv_2)\| \le l \|v_1 - v_2\| \ \ , \ \forall \ v_1, v_2 \in M$$

$$\|(I-P)(Gv_1 - Gv_2)\| \le \delta \|v_1 - v_2\| \ , \ \forall \ v_1, v_2 \in M$$

where $\delta < 1$. *We also assume that* $l \ge 1 - \delta$. *Then the compact* M *possesses a finite fractal dimension.*

2. THE MODEL

We begin by presenting PDE model to be considered. Let H be a separable Hilbert space and we consider the following second order in time equation

$$\begin{cases} u_{tt} + \delta u_t + A^\alpha u + g(u) A^\beta u + Lu = f \\ u(0) = u_0 \ \ , \ \ u_t(0) = u_1 \end{cases} \tag{1}$$

where A is a positive operator with discrete spectrum in H, M is a real function and L is a linear operator in H whose properties will be stablished below; f is a given bounded function with values in H, $g(u) = M\left(\left\|A^{\frac{\beta}{2}} u\right\|^2\right)$; α, δ and β are a positive real numbers.

The initial value problem (1) generalizes the boundary initial value problem

$$\begin{cases} u_{tt} + \delta u_t + \Delta^2 u + g(u)\Delta u + \xi \dfrac{\partial u}{\partial x_i} = f \\ u(x,t) = \Delta u(x,t) = 0 \ , \ x \in \partial\Omega \\ u(0) = u_0, \ \ u_t(0) = u_1, \ x \in \Omega \end{cases} \tag{2}$$

where $g(u) = \left(\zeta - \int_\Omega |\nabla u|^2 \, dx\right)$. This model describes nonlinear oscillations of a beam Ω which is located in a supersonic gas flow moving along the x_i- axis. The parameter ξ is determined by the velocity of the flow. The function u measure the beam deflection at the point x and the instant t.

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS

We consider the initial value problem (1) and we assume the following hypothesis

$(H1)$ $\alpha \ge 2\beta > 0$

$(H2)$ $u_0 \in D\left(A^{\frac{\alpha}{2}}\right)$, $u_1 \in H$, $f \in L^\infty(0,T;H)$

$(H3)$ $M \in C^1(\mathbb{R}^+)$ and

$$M(z) = \int_0^z M(\xi)\, d\xi \ge -az - b$$

where $0 \le a < \lambda_1$, $b \in \mathbb{R}$ and λ_1 is the first eigenvalue of the operator A.

$(H4)$ $L : D\left(A^{\frac{\alpha}{2}}\right) \to H$ is a linear operator defined on $D\left(A^{\frac{\alpha}{2}}\right)$ and satisfies the condition

$$\|Lu\| \le C \left\|A^{\frac{\alpha}{2}} u\right\| \ \ , \ \ \forall \ u \in D\left(A^{\frac{\alpha}{2}}\right)$$

We define the space

$$W^\alpha(0,T) = \left\{\varphi \in L^2\left(0,T; D\left(A^{\frac{\alpha}{2}}\right)\right); \ \varphi' \in L^2(0,T;H)\right\}$$

which is a Banach space with the norm

$$\|\varphi\|^2_{W^\alpha(0,T)} = \|\varphi\|^2_{L^2\left(0,T;D\left(A^{\frac{\alpha}{2}}\right)\right)} + \|\varphi'\|^2_{L^2(0,T;H)}$$

Definition. We say that the function $u \in W^\alpha(0,T)$ is a *weak solution* of problem (1) on the interval $[0,T]$ if $u(0) = u_0$ and the equation

$$-\int_0^T (u' + \delta u, \psi')\, dt + \int_0^T \left(A^{\frac{\alpha}{2}} u, A^{\frac{\alpha}{2}} \psi\right) dt \quad (3)$$
$$+\int_0^T M\left(\left\|A^{\frac{\beta}{2}} u(t)\right\|^2\right)\left(A^{\frac{\beta}{2}} u(t), A^{\frac{\beta}{2}} \psi(t)\right) dt$$
$$+\int_0^T (Lu(t), \psi(t))\, dt$$
$$= (u_1 + \delta u_0, \psi(0)) + \int_0^T (f(t), \psi(t))\, dt$$

holds for any function $\psi \in W^\alpha(0,T)$ such that $\psi(T) = 0$.

Using the compactness method (see Lions (1969)) we prove the following assertion on the existence and uniqueness of weak solutions to problem (1).

Theorem D. Assume that hypothesis $(H1) - (H3)$ hold and let T be a positive real number. Then the problem (1) has a unique weak solution u in the class

$$u \in C\left([0,T]; D\left(A^{\frac{\alpha}{2}}\right)\right) \cap C^1([0,T]; H) \quad (4)$$

and satisfies the energy equality

$$E(u(t), u'(t)) = E(u_0, u_1) + \quad (5)$$
$$+\int_0^t \left[-\delta \|u'(s)\|^2 + (-Lu(s) + f(s), u'(s))\right] ds$$

where the energy functional $E : H \times H \to \mathbb{R}$ is defined by

$$E(\varphi, \psi) = \frac{1}{2}\left[\left\|A^{\frac{\alpha}{2}}\varphi\right\|^2 + \|\psi\|^2 + \mathcal{M}\left(\left\|A^{\frac{\beta}{2}} u\right\|^2\right)\right] \quad (6)$$

In the stationary case $f(t) \equiv f$ we can define an evolutionary operator S_t of problem (1) in the space $H = D\left(A^{\frac{\alpha}{2}}\right) \times H$ by

$$S_t y = (u(t), u'(t)) \quad (7)$$

for $y = (u_0, u_1) \in H$, where u is a weak solution to problem (1) with initial conditions $y = (u_0, u_1)$.

Due to the uniqueness of weak solutions S_t satisfies the semigroup properties

$$S_t \circ S_r = S_{t+r} \quad , \quad S_0 = I \quad , \quad t, r \geq 0$$

4. THE MAIN RESULT

In addition to the hypothesis $(H1)-(H4)$ stated above we assume further that

$(H5)$ there exists positive constants γ, a_1, a_2 and a_3 such that

$$M(z) - a_1 \int_0^z M(s)\, ds \geq a_2 z^{1+\gamma} - a_3$$

$(H6)$ there exists $0 \leq \theta < \frac{\beta}{2}$ and $C > 0$ such that

$$\|Lu\| \leq C\left\|A^{\frac{\beta}{2}+\theta} u\right\| \quad , \quad \forall\, u \in D\left(A^{\frac{\alpha}{2}}\right)$$

$(H7)$ for some $\sigma > 0$ we have

$$f \in D(A^\sigma), \quad LD(A^\sigma) \subset D(A^\sigma)$$
$$\|A^\sigma Lu\| \leq C\left\|A^{\frac{\alpha}{2}} u\right\|$$

In order to establish our main result we give the following preliminary results:

Theorem 1. Suppose that the hypothesis $(H1) - (H3)$, $(H5)$ and $(H6)$ are fulfilled. Then the dynamical system $(\mathcal{H}, S_t)$ generated by (1) with $f(t) \equiv f \in H$ is dissipative.

Proof. The proof of this theorem is based in Theorem A. In this sense, is sufficient to show that there exists a functional $V : H \to \mathbb{R}$ which is bounded on the bounded sets of the space H, differentiable along the trajectories of system (1) and such that

$$V(y) \geq \rho \|y\|^2_{\mathcal{H}} - C_1 \quad (8)$$

$$\frac{d}{dt} V(S_t y) + \varepsilon V(S_t y) \leq C_2 \quad (9)$$

where ρ, $\varepsilon > 0$ and C_1,$C_2 \geq 0$ are constants.

Let

$$V(y) = E(y) + \nu \Phi(y) \quad (10)$$

where $y = (u_0, u_1) \in H$, E is the energy defined in (6), Φ is a functional defined by

$$\Phi(y) = (u_0, u_1) + \frac{\delta}{2}\left\|A^{\frac{\alpha}{2}} u\right\|^2$$

and ν is a parameter that will be chosen below.

Using elementar inequalities is easy to proof that the function V satisfies conditions (8) and (9).

Theorem 2. Suppose that the hypothesis $(H1) - (H3)$, $(H5) - (H7)$ are fulfilled. Then there exists a positively invariant bounded set K_σ in the space $H_\sigma = D\left(A^{\frac{\alpha}{2}+\sigma}\right) \times D(A^\sigma)$ which is closed in $\mathcal{H}$ and such that

$$\sup\{\mathrm{dist}_{\mathcal{H}}(S_t y, K_\sigma);\ y \in B\} \leq$$
$$\leq C \exp\left(-\frac{\delta}{4}(t - t_B)\right) \quad (11)$$

for any bounded set B in the space $\mathcal{H}$ and $t > t_B$.

Proof. Since the system $(\mathcal{H}, S_t)$ is dissipative by Theorem 1, there exists $R > 0$ such that

$$\|u'(t)\|^2 + \left\|A^{\frac{\alpha}{2}} u(t)\right\|^2 \leq R^2 \ , \tag{12}$$

for all $y \in B$ and $t \geq t_0 = t_0(B)$, where u is a weak solution of (1) with initial conditions $y = (u_0, u_1) \in B$, being B a bounded set of $\mathcal{H}$.

Denoting by U the semigroup generated by the homogeneous problem associated to (1) we can to prove that there exists $N_0 \geq 0$ such that

$$\|(I - \pi_N) U(t, \tau) \xi\|_{\mathcal{H}_\sigma} \leq C \|\xi\|_{\mathcal{H}_\sigma} \exp\left(-\frac{\delta}{4}(t - \tau)\right) \tag{13}$$

for $N \geq N_0$, $t \geq \tau \geq t_0$, and π_N is a orthogonal projection in $\mathcal{H}$ onto

$$\text{span}\left\{(w_k, 0), (0, w_k), \ k = 1, 2, \ ..., N\right\}$$

and $\{w_k\}_{k\in\mathbb{N}}$ is the orthonormal basis of the eigenvalues of the operator A in H.

Then following the ideas exposed in Temam (1988) we obtain a number R_σ depending on the radius of dissipativity R such that the vector

$$S_t y - (I - \pi_{N_0}) U(t, t_0) S_{t_0} y =$$

$$= \pi_{N_0} S_t y + (I - \pi_{N_0}) G(t, t_0; y)$$

lies in the ball $B_\sigma = \left\{y \ ; \ \|y\|_{\mathcal{H}_\sigma} \leq R_\sigma\right\}$ for $t \geq t_0$, where G is given by

$$G(t, t_0; y) = -\int_{t_0}^{t} U(t, s)(0, Lu(s) - f)\, ds$$

From (12) and (13) we can easily deduce that

$$\text{dist}_{\mathcal{H}}\ (S_t y, B_\sigma) \leq CR \exp\left(-\frac{\delta}{4}(t - t_0)\right)$$

Taking $K_\sigma = \gamma^+(B_\sigma) = \bigcup_{t\geq 0} S_t(B_\sigma)$ we can verify that all conditions of Theorem 2 are satisfied by K_σ.

Theorem 3. Suppose that the hypothesis $(H3)$,$(H5)$, $(H6)$ and $(H7)$ are fulfilled. Then the dynamical system $(\mathcal{H}, S_t)$ generated by problem (1) possesses a global attractor $\mathcal{A}$ of a finite fractal dimension. This attractor is a connected compact set in H and is bounded in the space $\mathcal{H}_\sigma = D\left(A^{\frac{\alpha}{2}+\sigma}\right) \times D(A^\sigma)$ where $\sigma > 0$ is defined by $(H7)$.

Proof. The existence of global attractor and the fact of that this attractor is a connected compact set in $H$and is bounded in the space $\mathcal{H}_\sigma = D\left(A^{\frac{\alpha}{2}+\sigma}\right) \times D(A^\sigma)$is consequence of Theorem B jointly with Theorems 1 and 2. Then we should prove only the finite dimensionality of the attractor. In order to obtain the finite dimensionality of the attractor $\mathcal{A}$ we will use the Theorem 1.3. With this aim we proceed in three steps:

Step 1. For any pair of semitrajectories $\{S_t y_j \ ; \ t \geq 0\}$, $j = 1, 2$, possessing the property $\|S_t y_j\| \leq R$ for all $t \geq 0$ the estimate

$$\|S_t y_1 - S_t y_2\|_{\mathcal{H}} \leq \exp(a_0 t) \|y_1 - y_2\|_{\mathcal{H}} \ , \quad t \geq 0$$

holds with the constant a_0 depending on R.

Step 2. For any y_1,$y_2 \in K_\sigma$ the inequality

$$\|(I - \pi_N)(S_t y_1 - S_t y_2)\|_{\mathcal{H}} \leq$$

$$\leq a_1 \left(1 + \frac{q_\sigma \exp(a_2 t)}{\lambda_{N+1}^\sigma}\right) \exp\left(-\frac{\delta}{4} t\right) \|y_1 - y_2\|_{\mathcal{H}} \ ,$$

hold for all $N \geq N_0$ being q_σ,a_1 and a_2 positive constants.

Step 3. Finally, let us choose t_0 and $N \geq N_0$ such that

$$a_1 \exp\left(-\frac{\delta}{4} t_0\right) = \frac{\mu}{2} \ , \ q_\sigma \lambda_{N+1}^{-\sigma} \exp(a_2 t_0) \leq 1 \ , \ \mu < 1$$

then the steps 1 and 2 enable us to state for y_1 and y_2 lie in the global attractor $\mathcal{A}$ that

$$\|S_{t_0} y_1 - S_t y_2\|_{\mathcal{H}} \leq l \|y_1 - y_2\|_{\mathcal{H}} \ , \text{ with } l = \exp(a_0 t_0)$$

and

$$\|(I - \pi_N)(S_{t_0} y_1 - S_{t_0} y_2)\|_{\mathcal{H}} \leq \mu \|y_1 - y_2\|_{\mathcal{H}}$$

Applying Theorem C with $M = A$, $G = S_{t_0}$, and $P = \pi_N$ we obtain the finite dimensionality of the global attractor A.

Acknowledgement. The author wishes to thank to Professor Richard Sanguino and the referees for his valuable comments and suggestions on this paper.

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