The Direct Boundary Element Method Applied to Modeling Differential and Integral Elastic Thin and Thick Plates Equations

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Abstract- Differential and integral elastic plates equations are frequently performed using numerical methods like the Finite Element Method (FEM) or the Boundary Element Method (BEM). One of the main differences between these techniques is a direct treatment of boundary value problem in BEM analysis. It must be pointed out that corner forces are introduced in the direct boundary integral equation when polygonal plates are studied using classical theory. This parameter appears as a consequence of classical plate hypothesis in which curvatures are related with the second derivative of the out-of-plane displacement and it is the necessary condition to reduce boundary variables. However, when thick plate theory is considered curvatures are not directly related with out-of-plane displacement derivative and no corner forces are introduced even in BEM analysis. Further, three boundary conditions should be satisfied in thick plate analysis rather than two of classical theory. This study intends to present results obtained from plate bending problems using classical and thick plate theories in order to understand the differences in plate behavior due the features above mentioned. The classical plate analysis will be performed using Danson's fundamental solutions and Reissner theory will be used in thick plate's analysis with Weeën's fundamental solution. The numerical implementation is carried out for continuous or discontinuous isoparametric linear elements. A classical example is solved to show the aim of this paper and the results are compared with those available in the literature.

KeyWord: Differential and Integral Equations, Thin and Thick Flat Plates, Boundary Elements.

O Método Direto dos Elementos de Contorno Aplicado a Equações Diferenciais e Integrais de Placas Finas e Espessas

Resumo- Equações diferenciais e integrais de placas elásticas são frequentemente resolvidas usando métodos numéricos como o Método dos Elementos Finitos (MEF) ou o Método dos Elementos de Contorno (MEC). Uma das principais diferenças entre essas técnicas é o tratamento do problema de valor de contorno na análise pelo MEC. Isso fica evidenciado quando forças de canto são introduzidas na equação integral de contorno quando placas poligonais são estudadas pela teoria clássica. Este parâmetro aparece como uma conseqüência das hipóteses clássicas da placa no qual as curvaturas são relacionadas com a segunda derivado dos deslocamentos fora do plano. Assim as forças de canto tornam-se uma condição necessária para a redução das variáveis de contorno. Entretanto, quando a teoria de placas espessas é considerada, as curvaturas não são diretamente relacionadas com a derivada do deslocamento fora do plano e as forças de canto desaparecem na análise pelo MEC. Além disso, três condições de contorno devem ser satisfeitas na placa espessa ao invés das duas da teoria clássica. Este estudo pretende apresentar resultados obtidos de problemas de flexão de placas usando a teoria clássica e a teoria de placas espessas no sentido de entender as diferenças no comportamento da placa devido aos fatores anteriormente mencionados. A análise da placa clássica será feita usando a solução fundamental de Danson e a teoria de Reissner será usada na análise da placa espessa juntamente com a solução fundamental de Weeën. A implementação numérica é feita usando elementos isoparamétricos contínuos e descontínuos. Um exemplo clássico é resolvido e os resultados são comparados com aqueles presentes na literatura.

Palavras-chave: Equações Diferenciais e Integrais, Placa Fina e Espessa, Elementos de Contorno.

1. INTRODUCTION

Thin plate analysis is widely used with BEM since Bézine (1978) or Danson (1979). It must be pointed out that thin plate analysis makes use of out-of-plane displacement derivatives in constitutive equations. Thus, two boundary condition needs to be satisfied to obtain one-value solution. On the other hand, thick plates are considered in BEM analysis since Weeën (1982) that introduced a formulation based in Reissner theory (1945). It must be pointed out that Reissner conceived his theory from an assumed stress distribution and got the corresponding strains using plane strain relations. Because of this assumption rotations and displacements represent weighted averages of actual displacements.
However, curvatures are not directly related with out-of-plane displacement derivative in the constitutive equations and no corner forces are introduced even in the boundary value problem of a polygonal plate. But, three boundary conditions should be satisfied in thick plate analysis rather than two of classical theory. The BEM formulations for thick plates had received several contributions along the years like the hypersingular formulation (Rashed, 1998) and a strategy to consider any thickness plate with Reissner theory (1945). Besides the numerical analysis, when a plate is analyzed for structural design purposes an engineer deals with a decision about the proper treatment of corner forces or if the corner forces should be included in the group of actual plate forces (Sanches, 2007). This paper intends to discuss this feature by mean of BEM analysis.

2. PLATE EQUATIONS

The faces of the plate are taken to be free from tangential traction but under normal pressures **q**, $\boldsymbol{\sigma_{\alpha 3}}$ is zero and $\boldsymbol{\sigma_{33}}$ is equal to **0.5q** in both faces. These expressions will be presented using the same notation used by Weeën (1982) or the Latin indices take values {1, 2, 3} and Greek indices in the range {1, 2}. A plate of uniform thickness is referred to midline coordinates $\mathbf{x_\alpha}$ and thickness coordinate $\mathbf{x_3}$.
The equilibrium equations for an infinitesimal plate element under a distributed transverse loading **q** are given by

$$M_{\alpha\beta,\beta} - Q_\alpha = 0 \qquad Q_{\alpha,\alpha} + q = 0 \tag{1}$$

The equilibrium equations are valid in both thin and thick plate's theory. The transverse shearing ($\mathbf{Q_\alpha}$), the bending and twisting moments ($\mathbf{M_{\alpha\beta}}$), all per unit of length, have similar definition in both theories. In thin plate theory, the constitutive relations in terms of out-of-plane displacement **w** are

$$M_{\alpha\beta} = -D.(1-\nu)\left(w,_{\alpha\beta} + \frac{\nu}{1-\nu} w,_{\gamma\gamma}\delta_{\alpha\beta}\right) \tag{2}$$

$$Q_\alpha = -D.w,_{\gamma\gamma\alpha}$$

$$D = \frac{Eh^3}{12\left(1-\nu^2\right)} \tag{3}$$

were **D** is the flexural rigidity, **E** is the Young modulus, **h** the thickness and $\boldsymbol{\nu}$ is the Poisson's ratio.
The Reissner's theory assumes a known stress distribution over the thickness and the used displacements are weighted averages of the actual displacements v_i in the reference coordinates directions

$$\sigma_{\alpha\beta} = \frac{12.x_3}{h^3} M_{\alpha\beta}; \quad \sigma_{\alpha 3} = \frac{3}{2.h}\left[1 - \left(\frac{2.x_3}{h}\right)^2\right] Q_\alpha$$

$$\sigma_{\alpha 3} = \frac{1}{4}\left(\frac{2.x_3}{h}\right)\left[3 - \left(\frac{2.x_3}{h}\right)^2\right] q \tag{4}$$

and

$$\phi_\alpha = \int_h \left(\frac{12.x_3}{h^3}.v_\alpha\right) dx_3$$

$$w = \int_h \frac{3}{2.h}\left[1 - \left(\frac{2.x_3}{h}.\right)^2\right] v_3 dx_3 \tag{5}$$

The constitutive relations obtained from a plane strain problem in an elastic body must be written in terms of displacements ($\boldsymbol{\phi_\alpha}$, **w**) and are given by

$$M_{\alpha\beta} = D\frac{1-\nu}{2}\left(\phi_{\alpha,\beta} + \phi_{\beta,\alpha} + \frac{2\nu}{1-\nu}\phi_{\gamma,\gamma}\delta_{\alpha\beta}\right) + \frac{\nu}{1-\nu}.\frac{q}{\lambda^2}\delta_{\alpha\beta}$$

$$Q_\alpha = D\frac{1-\nu}{2}\lambda^2\left(\phi_\alpha + w_{,\alpha}\right) \tag{6}$$

$\boldsymbol{\lambda}$ is a constant related to shear effect; it is equal to $\sqrt{10}/h$ in Reissner's theory.

Besides when the BEM analysis is performed, the direct boundary integral equation for a thin plate is

$$K_a w + \int_\Gamma \left[V_n^*.w - M_n^*.w,_n\right]d\Gamma + \sum_{i=1}^{N_c} R_{ci}^* w_{ci}$$

$$= \int_\Gamma \left[V_n.w^* - M_n.w^*,_n\right]d\Gamma + \sum_{i=1}^{N_c} R_{ci} w_{ci}^* + \iint q.w^* d\Omega \tag{7}$$

with,

$$R_{ci} = \left(M_{ns}^F - M_{ns}^B\right)_i \tag{8}$$

$\mathbf{K_a}$ depends only upon the geometry of the boundary at **x** and it equals **0.5** for the case of a continuous tangent; the number of corners is $\mathbf{N_c}$; $\mathbf{R_{ci}}$ is the corner reaction that is related to twisting moment $\mathbf{M_{ns}}$ in the forward (**F**) and backward (**B**) neighborhood with reference to the corner **i**. $\mathbf{w}^*$ is the Danson's fundamental solution to out-of-plane displacement due a unit point load

$$w^* = \frac{1}{8\pi D} r^2 \left(\ln r - \frac{1}{2}\right) \tag{9}$$

In thin plate formulation, when an uniform domain load was applied, there was used a conversion into a boundary integral

$$\iint q(y) w^*(x,y) d\Omega(y) =$$

$$q.\oint \frac{r(x,y)^3}{32\pi D}\left(Ln(r) - 0.75\right) r_\alpha n_\alpha(y) ds(y) \tag{10}$$

Using the concept of generalized displacements and forces due to Ween, the integral equation for thick plates may be written as

$$C_{ij}(x)u_j(x) + \oint T_{ij}^*(x,y)u_j(y)ds(y) = \oint U_{ij}^*(x,y)t_j(y)ds(y) +$$

$$+ \iint \left[U_{i3}^*(x,y) - U_{i\alpha,\alpha}^*(x,y)\frac{\nu}{(1-\nu)\lambda^2}\right] q(y) d\Omega(y) \tag{11}$$

where $\mathbf{u_\alpha}$ is $\boldsymbol{\phi_\alpha}$, $\mathbf{u_3}$ is **w**, $\mathbf{t_\alpha}$ is the product $\mathbf{M_{\alpha\beta}.n_\beta}$, $\mathbf{t_3}$ is the product $\mathbf{Q_\alpha.n_\alpha}$ and the differentials **ds(y)** and **dΩ(y)** denote boundary and domain differentials respectively.

The used fundamental solutions are given by

$$U_{\alpha\beta}^* = \frac{1}{2\pi D}\left\{\begin{array}{l} \frac{2}{1-\nu}\left[A(z) - B(z) r_{,\alpha} r_{,\beta}\right] \\ -\frac{1}{2}\delta_{\alpha\beta}\left(\ln z - \frac{1}{2}\right) - \frac{1}{2} r_{,\alpha} r_{,\beta} \end{array}\right\} \tag{12}$$

$$\begin{aligned} U_{\alpha 3}^* &= \frac{1}{2\pi D}\left[\frac{1}{2} r r_{,\alpha}\left(\ln z - \frac{1}{2}\right)\right]; \\ U_{3\alpha}^* &= -U_{\alpha 3}^*; \\ U_{33}^* &= -\frac{1}{2\pi D\lambda^2}\left[\frac{2}{1-\nu}\ln z - \frac{1}{4}z^2(\ln z - 1)\right] \end{aligned} \tag{13}$$

$$\begin{aligned} T_{i\alpha}^* &= D\frac{1-\nu}{2}\left(U_{i\alpha,\beta}^* + U_{i\beta,\alpha}^* + \frac{2\nu}{1-\nu}U_{i\gamma,\gamma}^*\delta_{\alpha\beta}\right)n_\beta; \\ T_{i3}^* &= D\frac{1-\nu}{2}\lambda^2\left(U_{i\alpha}^* + U_{i3,\alpha}^*\right)n_\alpha \end{aligned} \tag{14}$$

with,

$$\begin{aligned} z &= \lambda.r; \\ A(z) &= K_0(z) + \frac{1}{z}\left[K_1(z) - \frac{1}{z}\right]; \\ B(z) &= K_0(z) + \frac{2}{z}\left[K_1(z) - \frac{1}{z}\right] \end{aligned} \tag{15}$$

In thick plate formulation, when a uniform domain load was applied, there was used a Weeën's conversion into a boundary integral:

$$\iint \left[U_{i3}^*(x,y) - U_{i\alpha,\alpha}^*(x,y)\frac{\nu}{(1-\nu)\lambda^2}\right] q(y) d\Omega(y)$$

$$= q.\oint \left[R_{i,\alpha}^*(x,y) - U_{i\alpha}^*(x,y)\frac{\nu}{(1-\nu)\lambda^2}\right] n_\alpha(y) ds(y)$$

with

$$R_{i,\alpha\alpha} = U_{i3} \tag{16}$$

3. BEM FORMULATION AND IMPLEMENTATION

The numerical implementation of equations (7) and (11) used isoparametric linear elements that presented good results in Palermo (1992) for thin plate analysis and in Sanches (1999) and (2007) for thick plate analysis. All nodal parameters were placed at the ends of the element and when discontinuous boundary elements were used the collocation points were shifted to element inside at a distance equal to a quarter of element length. In thin plate implementation there is necessary to use a number of integral equations equal to a double of the nodes number plus an equation for each corner reaction of the problem. The equation (7) was written at boundary nodes and at external points. The external points were placed at a distance equal to a quarter of element length and in normal direction from each boundary node. In thick plate implementation there is necessary to use a number of integral equations equal to three times the nodes number. The equation (11) was written at external points for each one of the fundamental solutions (Sanches, 1999). The external points were placed at a distance equal to a quarter of element length and in normal direction from each boundary node. The algebraic equation is

$$K^i u_i + \sum_{j=1}^{Ne} H_{ij} u_j = \sum_{j=1}^{Ne} G_{ij} t_j \qquad (17)$$

The discrete boundary integral equation (17) describing the out-of-plane bending effect may be discretized as follows,

$$\begin{bmatrix} H^p_{11} & H^p_{12} \\ H^p_{21} & H^p_{22} \end{bmatrix} \begin{Bmatrix} w \\ w,_n \end{Bmatrix} = \begin{bmatrix} G^p_{11} & G^p_{12} \\ G^p_{21} & G^p_{22} \end{bmatrix} \begin{Bmatrix} V_n \\ M_n \end{Bmatrix} \qquad (18)$$

The upper indices p on the coefficient matrices H and G stand for thin plate mechanisms. The thin plate displacement normal to the x_1-x_2 plane is w and its derivative with respect to the boundary normal n is $w_{,n}$. The corresponding generalized forces are the shear forces V_n and the bending moment M_n. The boundaries described by equation (17) were discretized by rectilinear boundary elements described by linear shape functions.
Considering B_1 and B_2 the initial and final coordinates of the elements, the isoparametric element geometry may be expressed in terms of intrinsic coordinates, ς.

$$b(\varsigma) = B_1 \frac{1-\varsigma}{2} + B_2 \frac{1+\varsigma}{2} \qquad (19)$$

This same interpolation is used for the field variables of the boundary elements possessing no corners, leading to an isoparametrical formulation. For elements with corners the field variables were discretized by discontinuous elements. The corner nodes were displaced towards the interior by one fourth of the element length ($1/4L_e$). In this case, two integral equations were written for every boundary node. The collocation points were placed outside the plate domains, at distances d_1 and d_2, respectively. A final algebraic system is obtained once the equations are assembled and the prescribed boundary conditions applied. The solution of this algebraic system contains all unknown boundary quantities.

4. RESULTS

A simply supported square plate was loaded with a uniformly distributed load (**q**) over the domain that is equal to 1 kN/m^2 (Figure 1). The Poisson's ratio was adopted equal to zero. The plate side was 2 m long (a = 1m) and the thickness was 0,02 m. The boundary of the plate was divided into 12, 24 and 40 elements. Discontinuous boundary elements were used around the corners or two nodes at each corner.

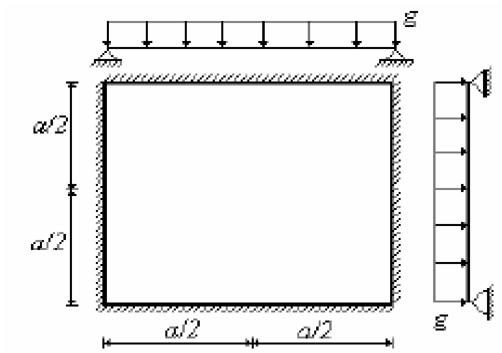

Figure 1: Square Plate loaded with a uniformly distributed load q.

The Figure 2 to present the first discretization adopted.

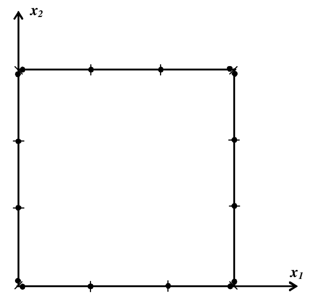

Figure 2: Discretization with 12 isoparametric boundary elements.

In the thin plate analysis, the maximum out-of-plane displacement (Figure 3) or the displacement at the central point was 0.27% different from the Timoshenko's solution, (1959). The bending moment M_{xx} in this point which is equal to M_{yy} was 0.11% different from Timoshenko (1959). In the thick plate analysis, the maximum out-of-plane displacement or the displacement at the central point was 0,30% different from, Timoshenko (1959).

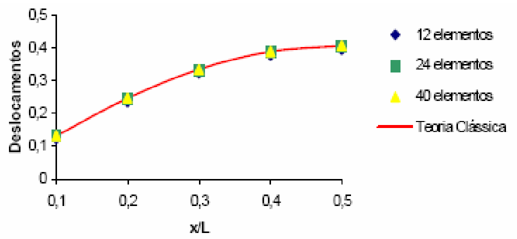

Figure 3: Thin plate out-of-plane displacement.

The plate bending moment M_{xx} in this point which is equal to M_{yy} was 0,12% different from Timoshenko (1959) (Figure 4).

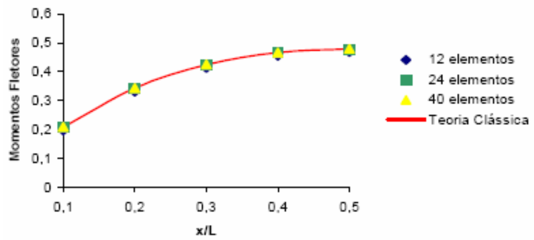

Figure 4: Thin plate bending moment M_{xx}.

Unless the solution presented in Timoshenko (1959) and Sanches (2007) was done according to thin plate theory, the agreement between the results in thick plate model is to note the behavior represented is close to a thin plate analysis.

5. CONCLUSION

It must be remembered that solutions for thin plates presented by Timoshenko for simply supported plates are due to Navier or Lévy. In Timoshenko (1959), there is a reference about explanations by Kelvin and Tait. These authors pointed out about local effect of forces due the couples which leaves the stress condition of the rest of plate unchanged. Navier or Lévy used a sum of trigonometric functions to represent the plate behavior. When Navier solution for a sinusoidal load is considered and the resultant of the Vertical Reaction is integrated over the plate boundary there was necessary to add corner forces to get the equilibrium with the external load. Thus, the final response from Navier is a sum of a trigonometric solution plus two singularity solution (i.e., the corner forces) on each side.

In this point, is necessary to introduce the BEM analysis which provides a proper treatment of singularities. Hence, structural engineers should take it into account and understand how the thin plate model responses on the vertical pressure distribution must be considered. But, if vertical pressure distribution is important in a problem the better analysis would be made whether the thick plate theory was used. Finally, the authors believed the obtained result must be improved in numerical analysis by mean of quadratic elements or special numerical treatments but the essential structural point of view included here will be the same.

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